Modeling Frequency-Dependent Boundaries as Digital Impedance Filters in FDTD and K-DWM Room Acoustics Simulations*

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A new method for modeling frequency-dependent boundaries in finite-difference time-domain (FDTD) and Kirchhoff variable digital waveguide mesh (K-DWM) room acoustics simulations is presented. The proposed approach allows the direct incorporation of a digital impedance filter (DIF) in the multidimensional (2D or 3D) FDTD boundary model of a locally reacting surface. An explicit boundary update equation is obtained by carefully constructing a suitable recursive formulation. The method is analyzed in terms of pressure wave reflectance for different wall impedance filters and angles of incidence. Results obtained from numerical experiments confirm the high accuracy of the proposed digital impedance filter boundary model, the reflectance of which matches locally reacting surface (LRS) theory closely. Furthermore a numerical boundary analysis (NBA) formula is provided as a technique for an analytic evaluation of the numerical reflectance of the proposed digital impedance filter boundary formulation.

0 INTRODUCTION

The finite-difference time-domain (FDTD) method is a wave-based numerical technique for solving partial differential equations, and as such can be applied to room acoustics simulations [1]. One of the key challenges in this area is to construct accurate and stable models of boundaries. The majority of FDTD boundary models available in the literature for nonstaggered grids, as well as those developed for the closely related digital waveguide mesh (DWM), are based on a 1D mesh termination. In this approach only one of the neighboring mesh nodes is employed for the computation of boundary node values; the idea is that for modeling a locally reacting surface (LRS), only two nodes (the boundary node and the adjacent node in the room interior space) are needed in order to derive the local velocity component normal to the wall. Various previous studies claim that the main advantage of the 1D approach is that it allows for direct modeling of a reflectance for a normal angle of incidence [2]–[5]. The first boundary model of this type has been derived from the original DWM equations in [2], and is based on terminating a 2D/3D mesh implemented in FDTD form—which in the DWM literature is often referred to as a Kirchhoff variable digital waveguide mesh (K-DWM)—with a 1D frequency-independent reflection coefficient. This approach can be extended to frequency-dependent boundaries by replacing the reflection coefficient with a digital filter [3]. For wave variable mesh implementations (W-DWM) this substitution is completely straightforward. For K-DWM modeling, a special adaptor, called “KW converter” [6], is required at each boundary junction to...
connect the Kirchhoff variable mesh to the wave-variable boundary [4], [5]. The change in wave equation dimensionality at the boundary is of course rather unphysical, and it has been shown in [7] for frequency-independent wall impedances that large errors can result in the reflectance phase and amplitude, particularly so for high angles of incidence and low reflectance values.

In [8] we have shown that a physically more correct FDTD model of frequency-independent boundaries is formulated by combining the boundary condition (expressed in terms of the wall impedance) with the multidimensional (2D or 3D) wave equation. In contrast to the 1D approach, the 2D/3D wave equation is now satisfied at a boundary, resulting in a numerical formulation that really behaves as a locally reacting surface. That is, the reflected sound wave approximates the theoretical LRS reflectance well resulting in a numerical formulation that really behaves as a locally reacting surface. That is, the reflected sound wave approximates the theoretical LRS reflectance well.

1 THE FDTD METHOD

The general objective of FDTD room acoustics modeling is the numerical solution of the wave equation that governs sound wave propagation in air, which reads

$$\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p$$ (1)

where $c$ denotes sound speed and $\nabla^2 p$ is given as

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}$$ (2)

for the 2D and 3D acoustic systems, respectively. FDTD schemes for numerical simulation of the wave equation are derived by approximating time and space derivatives with finite-difference operators. This technique is usually characterized by a regular spatial grid, the size of which depends on the sampling frequency. The nonstaggered, standard leapfrog rectilinear finite-difference formulation of the wave equation is obtained by applying centered difference operators to approximate the derivatives in Eq. (1). For example, the standard leapfrog scheme is obtained by applying the following approximations to discretize the 2D wave equation [10]:

$$\frac{\partial^2 p}{\partial x^2} = \frac{p_{i+1,m} - 2p_{i,m} + p_{i-1,m}}{X^2} + O(X^2)$$ (5)

$$\frac{\partial^2 p}{\partial y^2} = \frac{p_{i,m+1} - 2p_{i,m} + p_{i,m-1}}{Y^2} + O(Y^2)$$ (6)

where $X$ and $Y$ denote grid spacing, $T$ is the time step, $p_{i,m}^n$ is the pressure update variable, $n$ is a time index, and $l$ and $m$ denote spatial indices in the $x$ and $y$ directions. Assuming equal distances between grid points in all directions, the 2D and 3D discretized wave equations take the form of Eqs. (7) and (8):

$$p_{i,m}^{n+1} = \lambda^2 (p_{i+1,m}^n + p_{i-1,m}^n + p_{i,m+1}^n + p_{i,m-1}^n)$$
$$+ 2(1 - 2\lambda^2) p_{i,m}^n - p_{i,m}^{n-1}$$ (7)

$$p_{l,m,i}^{n+1} = \lambda^2 (p_{l+1,m,i}^n + p_{l-1,m,i}^n + p_{l,m+1,i}^n + p_{l,m-1,i}^n)$$
$$+ p_{l,m,i+1}^n + p_{l,m,i-1}^n + 2(1 - 3\lambda^2) p_{l,m,i}^n - p_{l,m,i}^{n-1}$$ (8)

where $\lambda = cT/JX$ denotes the Courant number and $l$, $m$, and $i$ denote spatial indexes in the $x$, $y$, and $z$ directions. The stability condition for the 2D standard leapfrog scheme amounts to $\lambda \leq 1/\sqrt{2}$ and for the 3D standard leapfrog scheme it is $\lambda \leq 1/\sqrt[3]{3}$ [11].

FDTD schemes generally suffer from numerical dispersion errors, that is, the numerical phase velocity differs from the theoretical phase velocity, and for 2D and 3D rooms this phenomenon is generally dependent on fre-
quency and the direction of propagation [12]. In order to minimize the dispersion error, the Courant number is usually chosen at the stability bound, that is, $\lambda = 1/\sqrt{2}$ and $\lambda = 1/\sqrt{3}$ for modeling sound wave propagation in 2D and 3D rooms, respectively.

For top values of the Courant number, the standard leapfrog rectilinear scheme is mathematically equivalent to the rectangular DWM [13]. The latter, when implemented in an FDTD form, is often referred to in the DWM literature as the Kirchhoff variable digital waveguide mesh (K-DWM), and its implementation is identical to the implementation of the standard leapfrog scheme.

2 GENERALIZED FREQUENCY-DEPENDENT BOUNDARY CONDITION

In general, real acoustic boundaries reflect waves in a frequency-dependent manner. That is, the reflected wave has a phase and amplitude that differ from those of the incident wave, and such changes diverge with frequency [14]. This phenomenon can be incorporated in an FDTD model with locally reacting surfaces by representing the specific acoustic wall impedance with a digital filter.

2.1 Locally Reacting Surfaces

A surface is called locally reacting when the normal component of the particle velocity at the surface of the boundary depends solely on the sound pressure in front of the boundary element and not on pressure in front of neighboring elements of the wall [14]. If we consider a sound wave traveling in a positive $x$ direction (that is, a right boundary), this can be expressed as

$$p = Z_w u_w.$$  (9)

where $p$ denotes pressure, $Z_w$ is the boundary impedance, and $u_w$ denotes the velocity component that is normal to the surface of the boundary. While the wave equation is derived from the principles of the conservation of mass and the conservation of momentum, only the latter one may be applied in isolation at the boundary [8]. For a boundary normal to the $x$ direction, the conservation of momentum equation reads

$$\frac{\partial p}{\partial x} = -p \frac{\partial u_w}{\partial t}.$$  (10)

where $\rho$ denotes the air density. Differentiating both sides of Eq. (9) with respect to time and using the resulting equation to substitute for $\partial u_w/\partial t$ in Eq. (10) yields the boundary condition for the right boundary in terms of pressure only (see [7] or [8] for details)

$$\frac{\partial p}{\partial t} = -c \frac{\partial p}{\partial x}.$$  (11)

where $c$ denotes the speed of sound and $\xi_w = Z_w / \rho c$ is the specific acoustic impedance.

The locally reacting surface (LRS) model provides a simple basis for modeling specular reflections, and the LRS concept is captured by two conditions. The first is the boundary condition itself. The second condition is that the multidimensional wave equation holds at the boundary. If these two conditions are met, the corresponding reflection coefficient—in this paper referred to as reflectance—is related to the specific acoustic impedance by (see [14])

$$R_0 = \frac{\xi_w \cos \theta - 1}{\xi_w \cos \theta + 1}.$$  (12)

where $\theta$ denotes the angle of incidence. From inspection of Eq. (12) it is clear that the specific acoustic impedance fully characterizes the reflective properties of the boundary for all angles of incidence.

2.2 Digital Impedance Filter

The digital impedance filter (DIF) can be obtained in various ways, some of which will be described in Section 6. One approach is to measure the wall reflectance at a certain angle of incidence, for example, normal to the wall. The averaged results of such measurements are usually available in the form of absorption coefficient values $\alpha$ provided for each octave band, which can be converted to octave-band reflection coefficients by $|R| = \sqrt{1 - \alpha}$ [14]. Note that such reflection coefficients do not include phase information. However, there are several other techniques that also incorporate the reflectance phase information (see, for example, [15], [16]). For both cases a normal-incidence digital reflectance filter $R_0(z)$ can then be designed to approximate the measured reflection data in all octave bands. Provided that this normal-incidence reflectance filter represents a passive boundary (namely, $|R_0(z)| \leq 1$), an appropriately physical (positive real) digital impedance filter can be calculated using the inverted form of Eq. (12) with $\theta = 0^\circ$,

$$\xi_w(z) = \frac{1 + R_0(z)}{1 - R_0(z)}.$$  (13)

The general specific impedance is thus expressed in this study as an IIR filter, where both nominator and denominator are of the same order $N$, equal to the highest order of the reflectance filter,

$$\xi_w(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}.$$  (14)

Note that an IIR impedance filter $\xi_w(z)$ results for reflectance filters of both FIR and IIR types and that the direct design of $\xi_w(z)$ from measured or modeled impedance data $\xi_w(\omega)$ can lead to unphysical representations of the boundary (that is, an impedance filter that is not positive real).

3 2D DIF MODEL

This section presents the FDTD formulation of the 2D DIF model for right and left boundaries in a rectilinear grid, an analogous K-DWM formulation, and the treatment for outer and inner corners.

3.1 Boundary Formulation

The boundary condition for a right boundary can be discretized by approximating the first-order derivatives in
the time and space domains of the continuous boundary condition given by Eq. (11) with centered operators, which yields
\[ \frac{P_{1,m}^{n+1} - P_{1,m}^{n-1}}{2T} = -\varepsilon \xi_m(z) \left( P_{1,m}^n - P_{1,m}^n \right). \] (15)

In order to derive the multidimensional FDTD formulation of a locally reacting surface, the discretized boundary equation [Eq. (15)] has to be combined with the discretized multidimensional wave equation [7], which for the 2D standard leapfrog scheme is given by Eq. (7). Since the aim is to incorporate a frequency-dependent digital wall impedance \( \xi_m(z) \), Eq. (15) is first transformed to the \( z \) domain,
\[ P_{1,m}(z - z^{-1}) = -\lambda \xi_m(z)(P_{1,m} - P_{1,m}) \] (16)
where \( P_{1,m} \) denotes the \( z \) transform of the discrete time-domain pressure \( p_{1,m}^n \). To enable incorporation of the filter in the finite-difference boundary condition, it is necessary to write the impedance filter in the following way:
\[ \xi_m(z) = \frac{b_0 + B(z)}{a_0 + A(z)} \] (17)
where filter nominator and denominator are given as
\[ B(z) = \sum_{i=1}^{N} b_i z^{-i} \] (18)
\[ A(z) = \sum_{i=1}^{N} a_i z^{-i}. \] (19)

Now substituting Eq. (17) into the discretized boundary condition given by Eq. (16) yields
\[ P_{1,m}(z - z^{-1}) = -\lambda \left( \frac{b_0 + B(z)}{a_0 + A(z)} \right)(P_{1,m} - P_{1,m}). \] (20)

In order to obtain a computable boundary condition, this condition is rewritten in terms of the intermediate value \( Y \),
\[ P_{1,m}(z - z^{-1}) = \lambda Y \] (21)
where
\[ Y = \frac{b_0 + B(z)}{a_0 + A(z)}(P_{1,m} - P_{1,m}). \] (22)

Eq. (22) can be treated as an input–output relationship \( Y(z) = \xi_m(z) X(z) \) with the transfer function given by the IIR impedance filter \( \xi_m(z) \) and the filter input given as \( X = P_{1,m} - P_{1,m} \). We may rewrite this explicitly in terms of the input pressure values as follows:
\[ Y = \frac{1}{a_0} \left( b_0 + B(z) \right)(P_{1,m} - P_{1,m}) - A(z)Y. \] (23)

This explicit filter formula is then split into two parts, the current filter input values and a new term \( G \), which is computed using past filter values only,
\[ Y = \frac{b_0}{a_0} (P_{1,m} - P_{1,m}) + \frac{G}{a_i} \] (24)
where
\[ G = B(z)(P_{1,m} - P_{1,m}) - A(z)Y. \] (25)

Next, substituting Eq. (24) into Eq. (21) yields
\[ P_{1,m}(z - z^{-1}) = \lambda \left[ \frac{b_0}{a_0}(P_{1,m} - P_{1,m}) + \frac{G}{a_i} \right]. \] (26)

Such a splitting is necessary in order to separate the current filter values \( P_{1,m} \) and \( P_{1,m} \) from the explicit filter equation. They both also appear in the discrete wave equation and hence are needed for updating the boundary node. Whereas the first value can be computed explicitly from the discrete wave equation, the second can only be updated from the boundary condition, since \( P_{1,m} \) is an extra node located beyond the boundary, as shown in Fig. 1(a), and thus no other acoustic laws apply at this node. In addition, the introduction of intermediate variables appears to be a strict necessity, since avoiding this [by multiplying both sides of Eq. (20) with \( a_0 + A(z) \) and working out a different set of update equations] leads to a system that is not structurally stable. (Instabilities can arise from numerical round-off errors in that case.)

The “ghost point,” which is defined as a mesh node that is lying outside the modeled space [10] [see Fig. 1(a)], can now be written explicitly as
\[ P_{1,m} = \frac{a_i}{a_0}(P_{1,m}^n - P_{1,m}^n) + \frac{G}{a_0} \] (27)
and its inverse \( z \) transform reads
\[ P_{1,m} = P_{1,m}^n + \frac{a_0}{a_i}(P_{1,m}^n - P_{1,m}^n) + \frac{G}{a_0}. \] (28)

Similarly, the inverse \( z \) transform of Eq. (24) is
\[ y^n = \frac{1}{a_0} \left[ b_0 (P_{1,m}^n - P_{1,m}^n) + g^n \right] \] (29)
where \( y^n \) is the filter output value at time step \( n \) and the intermediate value \( g \) is the inverse \( z \) transform of \( G \),
\[ g^n = \sum_{i=1}^{N} \left[ b_i (P_{1,m}^{n-i} - P_{1,m}^{n-i}) - a_i y^{n-i} \right]. \] (30)

Finally, the update formula for the boundary node \( p_{1,m} \) is obtained by substituting for the ghost point in the dis-
cretized 2D wave equation given by Eq. (7) with Eq. (28). Hence the final update formula for a boundary node is
\[ p_{lm}^{n+1} = \left[ \frac{\sqrt{4(2p_{lm}^n + p_{lm+1}^n + p_{lm-1}^n) + 2(1 - 2\lambda^2)p_{lm}^n}}{b_0^2} + \frac{\lambda^2 \lambda^2}{b_0^2} \right] \left( 1 + \frac{\lambda^2 \lambda^2}{b_0^2} \right)^{-1}. \]  
\[ \text{(31)} \]

This DIF boundary formulation requires updating the boundary node \( p_{lm} \), the ghost point \( p_{ls+1,m} \), and the filter output \( y \) values at each time step according to Eqs. (31), (28), and (29). One IIR filter is necessary for each boundary node, with the input to the IIR filter at node \((l, m)\) defined as \( x^n = p_{ls+1,m}^n - p_{ls+1,m}^{n+1} \), and the intermediate value \( g \) can be obtained directly from the impedance filter implementation as indicated with Eq. (30). Since the order of both the impedance filter numerator and denominator is \( g \), the filter is most efficiently implemented in canonical form.

### 3.2 Other Rectilinear-Grid Boundaries

The derivation of the formulas for the other boundaries in a rectilinear grid (boundaries on the left, upper, and lower sides) proceeds in the same manner, but involves changing the relevant indexes and changing the sign of the flow normal to the boundary where required [8]. For example, the final update formula for the boundary node of a left boundary, as depicted in Fig. 1(b), is
\[ p_{lm}^{n+1} = \left[ \frac{\sqrt{4(2p_{lm+1}^n + p_{lm}^n + p_{lm-1}^n) + 2(1 - 2\lambda^2)p_{lm}^n}}{b_0^2} + \frac{\lambda^2 \lambda^2}{b_0^2} \right] \left( 1 + \frac{\lambda^2 \lambda^2}{b_0^2} \right)^{-1}. \]  
\[ \text{(32)} \]

the ghost point \( p_{l+1,m} \) is computed according to
\[ p_{l+1,m}^n = p_{l+1,m}^{n+1} + \frac{a_0}{\lambda b_0} (p_{l+1,m}^{n+1} - p_{l+1,m}^{n+1}) + g^n \]  
\[ \text{(33)} \]

with the explicit filter difference equation updated according to
\[ y^n = \frac{1}{a_0} (b_0 x^n + g^n) \]  
\[ \text{(34)} \]

where the intermediate value \( g \) yields
\[ g^n = \sum_{i=1}^{N} (b_0 x^{n-i} + a_i y^{n-i}) \]  
\[ \text{(35)} \]

and the filter input \( x \) at time step \( n \) is defined as
\[ x^n = p_{l+1,m}^n - p_{l+1,m}^{n+1}. \]  
\[ \text{(36)} \]

### 3.3 K-DWM Implementation

It is well known that the standard leapfrog FDTD scheme with the Courant number set to its highest possible value is mathematically equivalent to the rectangular DWM model [6]. Hence the boundary models presented here and in [7], [9] are also directly applicable to K-DWM simulations. For instance, the formula for a right boundary node of the 2D rectangular K-DWM is obtained by setting \( \lambda = 1/\sqrt{2} \), which reduces Eq. (31) to
\[ p_{lm}^{n+1} = \left[ 2p_{lm}^n + p_{lm+1}^n + p_{lm-1}^n + \frac{g^n}{b_0^2} \right] + (1 - 4r)p_{lm}^{n-1}. \]  
\[ \text{(37)} \]

where \( r = b_0/(2b_0 + \sqrt{2}a_0) \), the ghost point is updated according to
\[ p_{l+1,m}^n = \left[ p_{l+1,m}^{n+1} + \frac{\lambda b_0}{\lambda b_0 + a_0} (p_{l+1,m}^{n+1} - p_{l+1,m}^{n+1}) + g^n \]  
\[ \text{(38)} \]

and filter equations are again computed from Eqs. (29) and (30). Note that setting \( \lambda \) at its top value is generally a good strategy for FDTD simulation of rooms, since it results in the most efficient implementation of the room interior as well as the boundary. The efficiency arises from the smallest numerical error for top values of the Courant number, as has been explained in Section 1.

### 3.4 Corners

The treatment of corners in FDTD modeling is crucial for the overall stability of the whole simulation, and therefore appropriate equations have to be formulated with great care. A starting point for the derivation of update formulas for corner nodes is to realize that a corner is considered a part of the modeled space, and hence the multidimensional wave equation has to be satisfied at the corner node [8]. In this case a top-right outer corner [depicted in Fig. 2(a)] has to satisfy the 2D standard leapfrog scheme equation, and the two ghost points are eliminated with the use of two independent boundary conditions in the \( x \) and \( y \) directions, namely [8],
\[ \frac{\partial p}{\partial t} = \frac{\partial}{\partial x} \left( \xi_x \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( \xi_y \frac{\partial p}{\partial y} \right) \]  
\[ \text{(39)} \]

where \( \xi_x \) and \( \xi_y \) denote digital impedance filters in the \( x \) and \( y \) directions. Substituting for the ghost points in a 2D wave equation with the discrete versions of these boundary conditions results in the following update formula for a top-right outer corner node:
\[ p_{lm}^{n+1} = \left[ \frac{\lambda}{\lambda} (2p_{lm+1}^n + 2p_{lm+1}^n + \frac{g^n}{b_x} + \frac{g^n}{b_y}) + 2(1 - 2\lambda^2)p_{lm}^n \right] + \frac{\lambda b_0}{\lambda b_0 + a_0} (p_{lm+1}^n - p_{lm+1}^{n+1}) + \frac{g^n}{b_0^2} \]  
\[ \text{(40)} \]
for which both ghost points should be updated according to the update formulas

\[
\begin{align*}
  p_{l+1,m}^n &= p_{l,m}^n + \frac{a_x}{\Delta b_x} (p_{l+1,m}^{n-1} - p_{l,m}^n) + \frac{g_x}{b_x} \\
  p_{l,m+1}^n &= p_{l+1,m}^n + \frac{a_y}{\Delta b_y} (p_{l,m+1}^{n-1} - p_{l,m}^n) + \frac{g_y}{b_y}
\end{align*}
\]

(41)

(42)

where \(g_x\) and \(g_y\) are computed from their respective filter implementations, \(b_x\) and \(b_y\) are the \(b_i\) values, and \(a_x\) and \(a_y\) are the \(a_i\) values for the \(x\)- and \(y\)-direction boundary conditions.

Regarding the treatment of inner corners—in the FDTD literature also referred to as reentrant corners—neither of the two boundary conditions given by Eqs. (39) applies [8]. Furthermore there are no ghost points to eliminate [see Fig. 2(b)], and therefore the discrete 2D wave equation given by Eq. (7) can be used as an update formula for an inner corner node [8].

4 3D DIF MODEL

The formulation of the 3D DIF model, including the treatment of corners and edges, is accomplished in a similar manner to the 2D case presented in Section 3.

4.1 Boundary Formulation

The derivation of the update formula for a right boundary relies again on combining the discrete wave equation with the discrete version of the boundary condition given by Eq. (11). Thus from the generalized frequency-dependent boundary condition we arrive at the 3D form of Eq. (28),

\[
\begin{align*}
  p_{l+1,m,i}^n &= p_{l-1,m,i}^n + \frac{a_0}{\Delta b_0} (p_{l-1,m}^{n-1} - p_{l,m}^n) + \frac{g}{b_0}
\end{align*}
\]

(43)

where \(l\), \(m\), and \(i\) denote spatial indexes in the \(x\), \(y\), and \(z\) directions. The digital impedance filter difference equations are again given by Eqs. (34) and (35), for which the filter input is now defined as \(s_i = p_{l-1,m,i}^n - p_{l+1,m,i}^n\). Concerning a discrete 3D wave equation, the standard leapfrog scheme is given by Eq. (8) [12]. Substituting for the ghost point in Eq. (8) with Eq. (43) yields the update formula for the right boundary node,

\[
\begin{align*}
  p_{l,m,i}^{n+1} &= \left[ 2 \lambda^2 (p_{l-1,m,i}^n + p_{l,m,i}^{n-1} + p_{l,m+1,i}^n + p_{l,m}^{n-1}) \\
  &+ 2(1 - 3 \lambda^2) p_{l,m,i}^n + \frac{\lambda^2}{b_0} s_i^\alpha + \frac{\lambda a_0}{b_0} - 1 \right] p_{l,m,i}^{n-1} \\
  &\times \left( 1 + \frac{\lambda a_0}{b_0} \right)^{-1}.
\end{align*}
\]

(44)

This 3D formulation requires computing the update formulas for the boundary node \(p_{l,m,i}\) and the ghost point \(p_{l+1,m,i}\) according to Eqs. (44) and (45). The digital impedance filter is updated according to Eqs. (34) and (35), where \(s_i = p_{l-1,m,i}^n - p_{l+1,m,i}^n\).

5 NUMERICAL BOUNDARY ANALYSIS

Numerical boundary analysis is an analytic method proposed in [9], [8] for the prediction of the numerical boundary reflectance. In comparison to the numerical experiments discussed in Section 6, this method has the advantage that results can be computed much faster and are free from numerical artifacts. In [8] both the analytic approach and the numerical approach to determining the numerical reflectance have been validated by finding a high level of agreement between the respective results.

In this section we briefly explain the main concept of this analytic technique and provide formulas for an exact prediction of the numerical reflectance of boundaries modeled using the DIF approach explained in Sections 3 and 4.

5.1 2D DIF Model

Consider a wall normal to the rectangular coordinate system in the \(x\)-\(y\) plane parallel to the \(y\) axis and located at \(x = 0\). The total sound pressure in the plane of a locally reacting surface can be derived by adding incident and reflected sound pressure values. For complex frequency \(s = \alpha + j \omega\), the discrete-domain pressure at the right boundary node is

\[
p_{l,m}^n = p_0 e^{-sT} e^{-j\omega x \sin \phi} (e^{-j\omega x \cos \phi} + \tilde{R} e^{j\omega x \cos \phi})
\]

(47)

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where \( p_0 \) is the incident wave amplitude, \( \hat{R} \) denotes the numerical reflectance, and \( k \) is the discrete-domain wavenumber that can be computed for any real frequency \( \omega \) from the dispersion relation for the 2D standard leapfrog scheme [11],

\[
\frac{1}{\lambda^2} \sin^2 \left( \frac{\omega T}{2} \right) = \sin^2 \left( \frac{X \hat{k} \cos \theta}{2} \right) + \sin^2 \left( \frac{X \hat{k} \sin \theta}{2} \right). \tag{48}
\]

Eq. (47) is readily transformed to the \( z \) domain by \( e^{\nu T} = z \) and denoting the \( z \) transform of \( p^{n}_{m} \) by \( P^{n}_{m} \). We may also use Eq. (47) as a basis for substituting all the other pressure variables in the boundary update equation. Again applying \( z \) transforms, some examples of these substitutions are

\[
p^{n}_{m} \rightarrow P^{n}_{m}, \tag{49}
\]

\[
p^{n+1}_{m} \rightarrow z P^{n}_{m}, \tag{50}
\]

\[
p^{n-1}_{m} \rightarrow z^{-1} P^{n}_{m}, \tag{51}
\]

\[
p^{n}_{l-1,m} \rightarrow p^{n}_{l-1,m} e^{-j\delta X \sin \theta} [e^{-j(\ell - 1) X \cos \theta} + \hat{R} e^{j(\ell - 1) X \cos \theta}]. \tag{52}
\]

The variable \( \nu \) in Eq. (31) is substituted by a reformulated form of Eq. (25),

\[
G = [R(z) - A(z)\xi_\omega(z)](P^{n}_{l-1,m} - P^{n+1}_{l+1,m})
= H_{lw}(P^{n}_{l-1,m} - P^{n+1}_{l+1,m}). \tag{53}
\]

The NBA formula for the proposed DIF boundary model is then obtained by setting \( \ell = 0 \) (which corresponds to \( x = 0 \) at a boundary) and solving for the numerical reflectance \( \hat{R} \), which yields

\[
\hat{R}_d(z) = - \left\{ \left( 1 + \frac{\lambda a_0}{b_0} \right) z - \lambda^2 (2C + D + D') \right. \right.
\]
\[
+ 2(1 - \lambda^2) + \frac{\lambda^2}{b_0} (C^{-1} - C) H_{lw}(z) + \left[ 1 - \frac{\lambda b_0}{b_0} \right] z \left. \right\}
\]
\[
\times \left[ \left( 1 + \frac{\lambda a_0}{b_0} \right) z - \lambda^2 (2C^{-1} + D + D') + 2(1 - \lambda^2) \right] \right. \right.
\]
\[
+ \frac{\lambda^2}{b_0} (C - C^{-1}) H_{lw}(z) + \left[ 1 - \frac{\lambda a_0}{b_0} \right] z \left. \right\}^{-1} \tag{54}
\]

where \( C = e^{i\delta \hat{k} X \cos \theta} \) and \( D = e^{i\delta \hat{k} X \sin \theta} \). Note that Eq. (54) follows directly from numerical boundary analysis applied to Eq. (31). However, since an analysis technique does not have to result in a computable formula in the form of Eq. (31), Eq. (54) can be further rewritten in a simpler equivalent form as

\[
\hat{R}_d(z) = - \left\{ \left( 1 + \frac{\lambda}{\xi_\omega(z)} \right) z - \lambda^2 (2C + \lambda^2 (D + D')) \right. \right.
\]
\[
+ 2(1 - \lambda^2) + \left[ 1 - \frac{\lambda}{\xi_\omega(z)} \right] z \left. \right\}
\]
\[
\times \left\{ \left( 1 + \frac{\lambda}{\xi_\omega(z)} \right) z - \lambda^2 (2C^{-1} + \lambda^2 (D + D')) \right. \right.
\]
\[
+ 2(1 - \lambda^2) + \left[ 1 - \frac{\lambda}{\xi_\omega(z)} \right] z \left. \right\}^{-1}. \tag{55}
\]

Note that the substitution of discrete-domain pressure values in this procedure is valid only when the 2D (or 3D) discretized wave equation is imposed on the boundary node, which means that it is connected to adjacent nodes on the boundary [see also Fig. 11(a)]. Hence this method is not suitable for predicting the numerical reflectance of a 1D boundary formulation in a 2D/3D context, which includes all DWM boundary models of LRS type available in the literature.

The NBA formula can also be used to prove the stability of the DIF boundary model by showing that the boundary represents a passive termination. Passivity of an FDTD boundary formulation is a sufficient but not necessary condition for numerical stability and guarantees that all of the system’s internal modes are stable [17]. Furthermore such a passivity proof is similar to the standard procedure of proving the stability of FDTD schemes with the use of von Neumann analysis [11] or the Gustaffson-Kreiss-Sundström-Osher method for checking the stability of boundary conditions for finite-difference schemes [18].

First the 2D NBA formula given by Eq. (55) can be reformulated as

\[
\hat{R}_d(z) = - \frac{Q - 2\lambda^2 C}{Q - 2\lambda^2 C^{-1}} \tag{56}
\]

where for any numerical wavenumber such that \( -\pi/X \leq \hat{k} \leq \pi/X \) the new variable \( Q \) can be expressed as

\[
Q = \left[ 2 \cos(\omega T) - 2\lambda^2 \cos(\delta \hat{k} X \sin \theta) + 2(1 - 2\lambda^2) \right]
+ j \left[ \frac{2\lambda}{\xi_\omega(z)} \sin(\omega T) \right]. \tag{57}
\]

For any frequency the digital impedance filter response is complex-valued, that is, we may write \( \xi_\omega(z) = a_w + j b_w \). Its multiplicative reciprocal is

\[
\frac{1}{\xi_\omega(z)} = \frac{a_w}{a_w^2 + b_w^2} - j \frac{b_w}{a_w^2 + b_w^2} \tag{58}
\]

which now defines the variable \( Q \) as

\[
Q = \left[ 2 \cos(\omega T) - 2\lambda^2 \cos(\delta \hat{k} X \sin \theta) + 2(1 - 2\lambda^2) \right] \frac{2\lambda a_w}{a_w^2 + b_w^2} \sin(\omega T) + j \left[ \frac{2\lambda b_w}{a_w^2 + b_w^2} \sin(\omega T) \right]. \tag{59}
\]

The boundary model is passive when \( |\hat{R}_d(z)| \leq 1 \), which guarantees that there are no growing solutions of the system with respect to time. From Eq. (56) this condition can be written as

\[
|Q - 2\lambda^2 C| \leq |Q - 2\lambda^2 C^{-1}|. \tag{60}
\]

Taking the square of both the left-hand and right-hand sides of Eq. (60) yields

\[
|Q - C|^2 = \left| \text{Re}(Q) - 2\lambda^2 \cos(\delta \hat{k} X \cos \theta) \right|^2 + \left| \text{Im}(Q) - 2\lambda^2 \sin(\delta \hat{k} X \cos \theta) \right|^2. \tag{61}
\]
and

\[ |Q - C^{-1}|^2 = |\text{Re}\{Q\} - 2\lambda^2 \cos(\hat{k}X \cos\theta)|^2 + |\text{Im}\{Q\} + 2\lambda^2 \sin(\hat{k}X \cos\theta)|^2 \]  

(62)

where \( \text{Re}\{Q\} \) and \( \text{Im}\{Q\} \) denote the real and imaginary parts of \( Q \). Inserting Eqs. (61) and (62) into Eq. (60) and applying further algebraic manipulations yields the following condition:

\[
-2\lambda \frac{a_w}{a'_w + b'_w} \sin(\omega T) \sin(\hat{k}X \cos\theta) 
\leq 2\lambda \frac{a_w}{a'_w + b'_w} \sin(\omega T) \sin(\hat{k}X \cos\theta).
\]

(63)

Since \( \sin(\omega T) \geq 0 \) for frequencies up to Nyquist, \( \lambda \geq 0 \), and \( \sin(\hat{k}X \cos\theta) \geq 0 \) is satisfied for all angles \(-\pi/2 < \theta \leq \pi/2\), Eq. (63) can finally be reduced to

\[
a_w \geq 0.
\]

(64)

Thus the effective numerical reflectance is passive provided that the real part of the digital wall impedance is nonnegative. This is ensured for any impedance filter that has been calculated from Eq. (13) using a digital normal-incidence reflection filter for which \(|R_w(z)| \leq 1\). It also follows directly from the definition of a positive real impedance. Since the medium itself is lossless (none of the FDTD schemes used cause numerical attenuation), it follows that for any \( \lambda \) the simulation as a whole is always stable.

### 5.2 3D DIF Model

If we consider a wall normal to the rectangular 3D system \((x, y, z)\) and located at \(x = 0\), the discretized total sound pressure in the standing wave in the right boundary plane can be expressed as

\[
p_{z,\phi}^n = p_{0} e^{j\hat{k}X \cos\theta \cos\phi} e^{-j\hat{k}X \sin\theta \sin\phi} e^{-j\omega T} + \hat{R} e^{j\hat{k}X \cos\theta \cos\phi} e^{-j\omega T}.
\]

(65)

where \( \theta \) and \( \phi \) are the azimuth and elevation angles, respectively, and the discrete wavenumber \( \hat{k} \) can be obtained from the dispersion relation for the 3D standard leapfrog scheme, which reads [11]

\[
\frac{1}{\lambda^2} \sin^2\left(\frac{\omega T}{2}\right) = \left[ \sin^2\left(\frac{\hat{k}X \cos\theta \cos\phi}{2}\right) + \sin^2\left(\frac{\hat{k}X \sin\theta \sin\phi}{2}\right) + \sin^2\left(\frac{\hat{k}X \sin\phi}{2}\right) \right].
\]

(66)

Next, transforming Eq. (65) to the \(z\) domain and writing out the formulas for the pressure variables present in the boundary update equation given by Eq. (44), the 3D numerical boundary analysis formula results,

\[
\hat{R}_{w,\phi}(z) = -\left\{ \left( 1 + \frac{\lambda_0^2}{b_0^2} \right) z - [\lambda^2(2C + D + D^{-1} + E + E^{-1})
+ 2(1 - 3\lambda^2)] + \left( 1 - \frac{\lambda_0^2}{b_0^2} \right) z^{-1} + \frac{\lambda^2}{b_0^2}(C^{-1} - CH_w(z)) \right\]\n\times \left\{ \left( 1 + \frac{\lambda_0^2}{b_0^2} \right) z - [\lambda^2(2C^{-1} + D + D^{-1} + E + E^{-1})
+ 2(1 - 3\lambda^2)] + \left( 1 - \frac{\lambda_0^2}{b_0^2} \right) z^{-1} + \frac{\lambda^2}{b_0^2}(C - C^{-1})H_w(z) \right\}^{-1}\right\}^{1/2}.
\]

(67)

where \( C = e^{j\hat{k}X \cos\theta \cos\phi}, D = e^{j\hat{k}X \sin\theta \sin\phi}, \) and \( E = e^{j\hat{k}X \sin\phi} \). Concerning the passivity of the 3D DIF model, we can again apply the same procedure as for the 2D case, which after simple manipulations leads to Eq. (60), where \( Q \) now has more components. However, it is straightforward to show that Eq. (60) is again satisfied for all possible wavenumbers and angles of incidence, and hence the 3D DIF model is always stable.

### 6 NUMERICAL EXPERIMENTS

#### 6.1 Test Setup

For an investigation of the proposed numerical boundary formulation, the following procedure was designed. For the 2D room interior implementation a fourth-order accurate compact implicit scheme [19] with 1800×1400 mesh nodes was used, for which the Courant number was set to \( \lambda = 1/\sqrt{2} \) for grid spacing consistency with the boundary model. The size of the room and the simulation time (2000 samples at a sample rate of 4kHz) were set in such a way that only the reflections from the boundary investigated could reach a receiver position. Furthermore the simulation time was sufficiently long so that the whole wavelet could reach a receiver in order to obtain the most realistic reflectance plots for the frequency-domain analysis. Each test consisted of two simulations, the setup of which is depicted in Fig. 3, where a sharp impulse was injected into a mesh point. In each test the source position was chosen so that a constant distance of 400 grid points

![Fig. 3. Schematic depiction of simulation setup used in numerical experiments.](image-url)
from the center of the investigated wall was obtained and the incident waves at the angles of incidence $\theta = 0^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ$ resulted. The reflected signal $x_f$ was measured at a receiver position located at the same distance from the center of the wall as a source in the first simulation; $x_f$ inevitably included the “direct sound.” In the second simulation the wall was removed, the size of the modeled space increased by a factor of 2 to 1800×2800, and two signals were measured—the direct sound $x_d$ at the receiver position and the free-field signal $x_r$ at the mirror location of the receiver. The isolated reflected signal $x_r$ was obtained from $x_r = x_f - x_d$. There was an exception for $\theta = 60^\circ$ and $\theta = 75^\circ$, where in order to maintain a flatter wavefront, the size of the modeled space was increased to 2800×2100, the distance from the source was increased to 700 points, and the simulation time was extended to 3500 samples.

The experimentally determined numerical reflectance was defined as the frequency-domain deconvolution of $x_r$ and $x_d$. Furthermore all the measured signals were windowed with the right half of the Hanning window before applying Fourier transforms in order to reduce signal truncation errors.

6.2 1D Boundary Model

The 2D impedance filter boundary model proposed in this paper is compared with the 1D boundary model, which constitutes a typical termination of 2D and 3D acoustic spaces in the DWM room acoustics. This 1D FDTD/K-DWM boundary model is given by [2], [3]

$$p_{lm}^{n+1} = [1 + R_0]p_{l-1,m}^n - R_0p_{l,m}^{n-1} \quad (68)$$

where $R_0$ denotes the wall reflectance at normal incidence. By replacing $R_0$ with a digital filter $R_n(z)$, this formulation becomes mathematically equivalent to the 1D termination of a K-DWM model found in [4], [5].

6.3 Impedance Filter Design

In this section a brief description of the techniques used for the design of the impedance filters is given. In all cases the filters were designed to approximate both phase and amplitude of a particular wall type.

6.3.1 Fibrous-Material-Layer Boundary

The first boundary type under investigation was a layer of fibrous material stretched on a rigid wall. The analytic formulas for the characteristic impedance $Z_w(\omega)$ and the propagation constant $\Gamma(\omega)$ of the material provided in [16] were used as a basis to express the (continuous-domain) specific wall impedance $\xi_w(\omega)$ as follows:

$$\xi_w(\omega) = \frac{Z_w(\omega)}{\rho c} \coth[\Gamma(\omega)d] \quad (69)$$

where $d$ is the layer thickness and $\rho$ is the air density. Both $\Gamma$ and $Z_0$ depend on the flow resistivity $\sigma$. Hence the boundary is characterized by the two parameters $d$ and $\sigma$. This type of formulation was originally provided in [15] and later improved on in [16].

6.3.2 Mechanical-Impedance Boundary

A second boundary type was investigated, starting from a continuous-domain mechanical impedance $Z_w(\omega) = \rho c E_w(\omega)$, where $\rho$ denotes the mass per unit area, $E_0$ denotes the spring constant, and $\omega$ is the Laplace frequency variable. This impedance was mapped to the discrete domain using 1) the bilinear transform and 2) the impulse invariant method (IIM), resulting in two different digital impedance filters of second order to be investigated.

6.3.3 Low-Pass Reflectance Boundary

Finally we also designed a first-order digital low-pass reflectance filter by applying the IIM to the reflectance transfer function $R(z) = g/(s + \alpha)$, where $g$ is now a gain factor and $\alpha$ denotes a filter design parameter. The corresponding impedance filter is then obtained with Eq. (13).

6.4 Results of the 2D DIF Model

In this section the magnitudes of the experimentally obtained reflectance are analyzed in the frequency domain. In general figures show the reflectance of the 2D DIF model proposed in this paper (black dashed lines), the 1D boundary model that connects reflectance filters to the FDTD mesh using KW converters (black solid lines), and the theoretical reflectance (grey solid lines) up to a quarter of the sample rate. In order to focus on the FDTD modeling, the initial filter design approximation error was left out of the comparisons. That is, the theoretical reflectance was taken as the digital reflection for a given digital impedance filter, $R(z) = \xi_w(z) \cos \theta + 1$.

Fig. 4 illustrates the reflectance of the fibrous-material layer, where the thickness of the fibers was $d = 0.04$ m and the flow resistivity was assigned an example value of $\sigma = 100$ Ns m$^{-4}$. The DIF boundary model “follows” the theoretical value quite well for all angles of incidence. In particular, the numerical reflectance adheres perfectly to the theoretical reflection for low frequencies at all angles of incidence. Furthermore a perfect match is visible for $\theta = 45^\circ$, for which the leapfrog scheme has no dispersion. The numerical error is the largest at normal incidence, which coincides with the fact that the numerical dispersion error for the leapfrog scheme is the strongest in the axial directions.

1The IIM was actually applied to the specific wall admittance, and the resulting digital filter was then inverted to give the digital impedance. This indirect discretization is necessary since the IIM is not suited to mapping high-pass analog filters to the discrete domain.

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In comparison, the 1D boundary model does not adhere well to the theory for most angles of incidence. Although the numerical reflectance is on average at the correct level, there is a substantial misalignment along the frequency axis, that is, the minima and maxima are positioned wrongly for all angles of incidence.

Note that the jumps near $\omega = 0$ in the reflectance obtained with numerical experiments in all graphs, particularly visible at high angles of incidence ($\theta = 75^\circ$), are artifacts due only to the wavefront not being perfectly flat [9]. As illustrated in Fig. 5, the numerical boundary analysis confirms that a sudden jump near $\omega = 0$ for the angle of incidence $\theta = 75^\circ$ does not in fact occur for the boundary models presented.

Regarding the mechanical impedance boundary, the parameters were set to $R/(\rho c) = 2$, $M/(\rho cT) = 6$, and $KT/(\rho c) = 2$, where $T$ is the time step. When the bilinear transform method was used to obtain an impedance filter, the simulations with the 1D boundary model were structurally unstable. No stability problems occurred though when using the DIF model, and indeed a good match with theory was observed for all incidences. When using the impulse invariant method, neither boundary model resulted in stability problems. However, the 1D boundary model now shows an enormous discrepancy with theory, whereas the DIF model adheres extremely well to the theory (see Fig. 6).

The low-pass reflectance results are depicted in Fig. 7. As can be seen, the 1D boundary model performs somewhat better in this case, but is still structurally different from the theoretical response, and not nearly as accurate as the proposed DIF boundary model. In addition, since the numerical reflectance of the 1D boundary model is not exact at $\omega = 0$, there is no scope for improvement in accuracy by upsampling.

These results show that there are fundamental problems with the 1D boundary model, and that these are entirely overcome by the proposed DIF model. Furthermore it is interesting to note that for the DIF model the perfect match always results for $\theta = 45^\circ$ up to $0.25f_r$ and a weaker match is observed for the normal incidence.

Finally the 2D numerical-boundary-analysis formula for the last two boundary filters at normal angle of incidence is illustrated in Fig. 8. The reflectance was obtained through numerical experiments with the setup described in Section 6.1, with the difference that the room interior was modeled with the standard 2D rectilinear scheme, the size of the simulated space was increased to 2800x2100 and 2800x4200, respectively, the simulation time was increased to 3700 samples, and the distance from the source to the center of the investigated wall was set to 100 mesh nodes. The perfect match of the numerically simulated and analytically predicted numerical reflectance proves that numerical boundary analysis predicts the numerical reflectance of the DIF model accurately.
6.5 Results of the 3D DIF Model

In this section the results obtained with the use of 3D numerical boundary analysis are presented. Similarly to the previous section, all figures illustrate the magnitude of the numerical reflectance of a 3D DIF boundary model (black dashed lines) compared with the theoretical reflection (grey solid lines) for up to the cutoff frequency for axial directions, which for the 3D leapfrog scheme amounts to $0.196 f_s$. The theoretical reflection used as a reference was again taken as the digital reflection for a given DIF, which for a 3D space yields

$$R_{\phi,\theta}(z) = \frac{\xi_{\text{inc}}(z) \cos \theta \cos \phi - 1}{\xi_{\text{inc}}(z) \cos \theta \cos \phi + 1}. \quad (71)$$

Furthermore, the same three DIFs described in Section 6.3 were tested.

The reflectance of the fibrous-material layer for a constant elevation angle of $\phi = 60^\circ$ and varying azimuth angles is illustrated in Fig. 9. The numerical reflectance of the DIF model exhibits a very good agreement with theory, in particular at low frequencies. Note that the elevation angle selected for Fig. 9 was chosen as an example angle; however, a similarly good match was observed for other incidences.

Fig. 10 illustrates low-pass reflectance of the DIF model for an elevation angle of $\phi = 35^\circ$. A perfect match between the numerical and theoretical reflectance occurs for $\theta = 45^\circ$, which coincides with the lack of numerical error of the standard leapfrog scheme in diagonal directions.

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Fig. 6. Numerical reflectance of mechanical boundary discretized using impulse invariant method, for various angles of incidence $\theta$. ––– 2D DIF boundary model; 1D filter model; theoretical reflectance.

Fig. 7. Numerical reflectance of low-pass reflectance boundary with $g = 0.85$, $\alpha = 0.4$, and various angles of incidence $\theta$. ––– reflectance of 2D DIF boundary model; reflectsance from 1D filter model; theoretical reflectance.
Note that in a 3D case the diagonal direction can be defined by the pair of angles $\phi = 35.3^\circ$ and $\theta = 45^\circ$. It can also be observed that numerical reflectance structurally adheres well to the theory for the remaining azimuth angles.

These results show that the 3D DIF boundary model yields an accurate numerical reflectance for any combination of azimuth and elevation angles, even for near-grazing incidences.

7 DISCUSSION

In the 1D boundary model the boundary node connects only to the nearest node in the room interior, as illustrated in Fig. 11(b). Hence at the boundary node the only direction along which information is available is normal incidence. In other words, there is a priori no angle information available that can be incorporated to ensure that the numerical boundary behaves as a locally reacting surface. The only angle at which it can be hoped that the numerical reflectance will coincide with the theoretical reflectance is normal incidence. However, as the results of the numerical experiments show, even that is not the case.

Some light can be shed on the inaccuracies of the 1D boundary model by viewing it from an FDTD perspective. As has been shown in [8], the 1D FDTD/K-DWM boundary formula given by Eq. (68) implies that the distance from the boundary node to the nearest neighboring node in the medium is smaller than distances between room interior nodes. As illustrated in Fig. 11(c), when updating an interior node that is located next to the boundary, the distance between this node and the boundary node is larger than the distance in the opposite direction, that is, from the boundary node to the same interior node when updating the boundary node. Thus the distance between the same two mesh nodes varies depending on which node is updated. This constitutes an ambiguity about the internode spacing at the boundary, probably causing the simulation locally to be consistent with neither the 1D nor the 2D wave equation. While it remains difficult to precisely analyze or predict the effects of this ambiguity, the numerical experiments in this paper indicate that severe problems (including instability and large errors) can occur as a result.

On the other hand, the 2D/3D FDTD DIF formulation interconnects the boundary nodes to both the adjacent node in the interior as well as the neighboring nodes at a boundary, thus implementing wave propagation along the wall surface, and obeying the (discretized) 2D/3D wave equation at the boundary. As such it behaves as a locally reacting surface. As with dispersion errors in the standard leapfrog scheme, the numerical error is most pronounced at the higher end of the frequency axis, and vanishes al-

![Fig. 8. Numerical reflectance of mechanical boundary and low-pass reflectance boundary for $\theta = 0^\circ$. —— reflectance of 2D DIF boundary model with standard 2D rectilinear scheme room interior implementation; —— reflectance predicted with numerical boundary analysis.](image)

![Fig. 9. Numerical reflectance of fibrous-material boundary for $d = 0.04$, $\sigma = 100$, constant elevation angle $\phi = 60^\circ$, and various azimuth angles $\theta$. —— 3D DIF boundary model; —— theoretical reflectance.](image)
together for diagonal directions. Indeed, as shown in Fig. 8, the numerical reflectance of the multidimensional DIF model can be precisely predicted using methods that are in essence the same as those for analyzing dispersion for FDTD schemes.

The numerical results structurally match the magnitude of the theoretical reflectance well for up to the lowest cutoff frequency of the leapfrog scheme, approximating the 2D/3D wave equation (that is, up to \(0.25f_s\) for the 2D and up to \(0.196f_s\) for the 3D boundary model). In particular the numerical results always adhere perfectly at low frequencies, which is a fundamentally useful property of numerical modeling as it allows the improvement in performance by upsampling.

Furthermore the 2D/3D DIF formulation includes the proper treatment of corners and edges. In the 1D boundary model the corner is effectively not regarded as part of the medium, and therefore outer corner nodes are effectively excluded from the simulation, as shown in Fig. 11(b). This simplification leads to spurious results when a sound wave reflects from corner terminations of acoustic spaces. For example, when a plane wave traveling in the \(x\) direction reflects off a corner, some wave energy is directed in the \(y\) direction. No such problems occur when using the 2D/3D DIF model.

8 CONCLUSIONS

In this paper we have proposed a novel method for constructing numerical formulations of generalized frequency-dependent boundary models of a locally reacting surface in FDTD and K-DWM room acoustics simulations by incorporating digital impedance filters (DIFs) in a computable manner. Compact formulations of DIF boundary models were presented, and a good match with the theoretical reflectance magnitude was observed for various impedance filters and all angles of incidence. Furthermore a full treatment of corners and boundary edges is provided.

The proposed DIF boundary formulation represents a significant improvement over the commonly used 1D boundary model, which combines FDTD room interior implementation with reflectance filters at boundaries using KW converters. Up to now these have been the only means of modeling generally complex boundaries in nonstaggered FDTD and K-DWM simulations of acoustic spaces. Results obtained with numerical experiments indicate that the numerical reflectance of such 1D models can differ wildly from theory, and may even lead to instability problems. It seems therefore justified to say that the use of 1D filter models with KW converters at boundaries should be avoided.

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Fig. 10. Numerical reflectance of low-pass reflectance boundary with \(g/H_{11505} = 0.85\), \(H_{9251}/H_{11505} = 0.4\), constant elevation angle \(\phi = 35^\circ\), and various azimuth angles \(\theta\). –– reflectance of 3D DIF boundary model; — theoretical reflectance.

Fig. 11. 2D and 1D terminations of 2D mesh structure (a) 2D boundary. (b) 1D boundary. (c) 1D ambiguity; — room interior nodes. — boundary nodes; and arrows point to updated nodes.
In addition, the performance of the proposed DIF boundary model can be predicted exactly using a numerical boundary analysis method. This is extremely useful, as it allows researchers and developers to investigate the behavior of the numerical boundary without having to run many time-consuming simulations.

9 REFERENCES


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